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Exponentially Slow Dynamics and Interfaces Intersecting the Boundary

Michał Kowalczyk

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

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1. INTRODUCTION

In this paper we are concerned with the equation

$$\begin{cases} \Delta u - \varepsilon^{-2} F'(u) - u_t = 0, & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where u is a scalar function and $F \in C^3$ is a double-well potential satisfying: $F(-1) = F(1)$ and $F'(-1) = F'(0) = F'(1) = 0$, $F''(\pm 1) = 1$, $F''(0) < 0$. Here $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ is a bounded domain.

In the context of materials science (1.1), known as the *Allen–Cahn equation* (see [8]) serves as a model for the motion of the antiphase boundary that separates two phases of crystalline solid. Here u represents the long range parameter, the function F represents the free energy per unit volume and its two wells correspond to two different stable material phases. The equilibrium order parameters are $u = \pm 1$ and the antiphase boundary is the interface between two regions, one with $u \approx 1$ and the other with $u \approx -1$. Observe that (1.1) differs from the usual form of the Allen–Cahn equation; for convenience we rescaled time here so that t in the original formulation corresponds to $\varepsilon^{-2} t$.

By using the method of matched asymptotic expansions one can formally derive that as $\varepsilon \to 0$ then $u \to 1$ in a subregion $\Omega^+ \subset \Omega$, $u \to -1$ in another subregion $\Omega^- \subset \Omega$ and that Ω^+, Ω^- are separated by a hypersurface (interface) Γ. As the matching condition for the inner and outer expansion one obtains the law of motion for the interface:

$$V = -H, \tag{1.2}$$

where V is the normal velocity of Γ and H is its mean curvature (see [32]).

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The formal derivation of the motion by mean curvature was first justified rigoruosly in [16, 17] and independently, from a different point of view in [14]. These works assume the existence of smooth solutions to (1.2)—a hypothesis which in general can be verified only over short time intervals.

The analysis of generalized (viscosity) solutions to (1.2) was carried out in [15, 20]; the relation between the Allen–Cahn equation and the viscosity solutions to (1.2) for closed hypersurfaces and layered initial conditions was rigorously established in [19]; for interfaces intersecting the boundary analogous results were obtained in [25]. In [31] the convergence of the Allen–Cahn equation to the motion by mean curvature was justified for arbitrary initial data. For more information on this problem we refer the reader to [10, 11, 20, 28, 30–32] and the references therein.

The main goal of the present paper is to understand the dynamics of (1.1) when (1.2) degenerates and fails to provide an information about the motion of the interfaces. We consider a class of domains such that it is possible to construct an infinite set of equilibria to the motion by mean curvature. In our setting this set of equilibria consists of planar interfaces (so that $H\equiv 0$) intersecting the boundary of Ω orthogonally.

Based on the intuition that the continuum of equilibria of (1.2) should correspond to an invariant set on the level of PDE we follow here the constructive method initiated by Fusco and Hale in [22] and further refined by Fusco and Alikakos in [4, 5]. The key ingredient of this method is to construct an appropriate Ansatz $\mathcal{M}$ of the set invariant with respect to the flow of (1.1) and then, by introducing local coordinates, describe the flow determined by the Allen–Cahn equation near this set. By analyzing the flow in the direction tangential to the approximate invariant set we obtain the geometric law of motion for the interfaces, which can be thought of as an analog of (1.2). Since the approximate invariant set is in our case a curve in a function space therefore, unlike the motion by mean curvature, this law of motion is an ODE. Moreover the curvature of the interface does not play a role here; instead the geometry of the domain far from the interface determines the speed of the motion. Furthermore we establish that this law of motion is valid as long as the solution remains close to $\mathcal{M}$. This is done by investigating the flow in the direction transversal to $\mathcal{M}$.

We remark that the formal method of finding such law through matching algebraic (in ε) terms of inner and outer asymptotic expansions fails here; in fact the motion of the interface is determined not by algebraic but by transcendentally small terms in ε. Consequently the interface moves at an exponentially slow (in ε) rate and the solutions of (1.1) remain close to $\mathcal{M}$ for exponentially long time.

The phenomenon of the superslow (exponentially slow) motion for the Allen–Cahn equation in one space dimension was first investigated formally by Neu [29]. The rigorous results were obtained in [22, 23, 2] and by a slightly different approach in [12, 13]. For the Cahn–Hilliard equation analogous results were established in [1] (one-layered solutions) and [9] (multilayered solutions). Finally for the multidimensional Cahn–Hilliard equation Alikakos and Fusco [4, 5] showed superslow motion of near circular interfaces (see also [3]). Formal derivation of the exponentially slow motion for the non-local Allen–Cahn equation was carried out in [33].

We will now describe our special setting. In what follows we assume that $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ is a bounded, open domain that consists of a central, cylindrical part with two attachements on its sides. More precisely for the rest of this paper Ω will be a domain satisfying the following hypothesis:

(D1) $\partial\Omega \in C^{1,\alpha}$, $\alpha > 0$.

(D2) By $n = n(x)$, $x \in \partial\Omega$ we denote the unit outward normal. We assume that $n(x) = [0, n'(x)]$, for $x = (x_1, x')$, $0 < x_1 < 1$ where $x' = [x_2, ..., x_n]$, $n' = [n_2, ..., n_N]$.

(D3) There exists $a > 0$ and diffeomorphisms y^+, y^- such that

$$y^+ \colon (1, 1+a) \times S^{N-2} \to \partial\Omega \cap \{1 + a > x_1 > 1\},$$

$$y^- \colon (-a, 0) \times S^{N-2} \to \partial\Omega \cap \{-a < x_1 < 0\}.$$

(D4) Let $|\partial y^\pm / \partial(x_1, \theta)|$ be the jacobian of $y^\pm$ and let

$$\phi^\pm(x_1) = \int_{S^{N-2}} n_1(x_1, \theta) \left| \frac{\partial y^\pm}{\partial(x_1, \theta)} \right| d\theta.$$

We assume that there exist $K^\pm \neq 0$ and $\alpha^\pm \geqslant \alpha$ such that

$$\phi^-(x_1) = -(-x_1)^{\alpha^-} K^- + o(|x_1|^{\alpha^-}), \qquad x_1 \to 0^-,$$

$$\phi^+(x_1) = -(x_1 - 1)^{\alpha^+} K^+ + o(|x_1 - 1|^{\alpha^+}), \qquad x_1 \to 1^+.$$

Observe that besides (D1) we do not make any assumptions about the shape of $\partial\Omega$ far, to the left or right, from the cross sections $\{x_1 = 0\}$, $\{x_1 = 1\}$.

It is clear that for any ξ, $0 < \xi < 1$ the planar interface $\{x_1 = \xi\}$ satisfies $H \equiv 0$ and, by (D2) meets $\partial\Omega$ orthogonally, hence it is an equilibrium solution to (1.2). We will now construct a family of approximate solutions to (1.1) with the property that the zero level set of each function in this family is equal to $\{x_1 = \xi\}$. We will parametrize this family by ξ so that it can be thought of as one dimensional manifold in the function space.

By U we shall denote the heteroclinic solution to

$$U_{\eta\eta} - F'(U) = 0, \qquad U(\pm\infty) = \pm 1, \qquad U(0) = 0.$$

We set

$$\begin{cases} u^{\xi}(x) = U\left(\dfrac{x_1-\xi}{\varepsilon}\right), & x \in \Omega, \quad 0<\xi<1, \\ \mathscr{M} = \{u^{\xi} \mid 0<\xi<1\}. \end{cases} \tag{1.3}$$

The key observation for further analysis is that for each $\xi \in (0, 1)$ we have

$$\begin{cases} \Delta u^{\xi} - \varepsilon^{-2} F'(u^{\xi}) = 0, & \text{in } \Omega, \\ \dfrac{\partial u^{\xi}}{\partial n} = 0, & \text{on} \quad \partial\Omega \cap \{x \mid 0 \leqslant x_1 \leqslant 1\}, \end{cases} \tag{1.4a}$$

which upon differentiating with respect to ξ yields

$$\Delta u^{\xi}_{\xi} - \varepsilon^{-2} F''(u^{\xi})\, u^{\xi}_{\xi} = 0, \qquad \text{in } \Omega. \tag{1.4b}$$

We also have the estimates

$$\begin{gathered} |u^{\xi}_{\xi}(x)| = |u^{\xi}_{x_1}(x)| = O(\varepsilon^{-1} e^{-|x_1-\xi|/\varepsilon}), \\ \left| \dfrac{\partial u^{\xi}}{\partial n}\Big|_{\partial\Omega} \right| \leqslant C\varepsilon^{-1} \max\{e^{-\xi/\varepsilon}, e^{-(1-\xi)/\varepsilon}\}, \end{gathered} \tag{1.5}$$

with similiar formulas holding for the higher order derivatives of u^{ξ}.

For the remainder of this paper we fix δ, a small number, independent on ε. Under the assumption $\delta < \xi < 1-\delta$ the estimates in (1.5) are uniform in ξ. We set

$$\mathscr{N}(\delta) = \{u \in L^2(\Omega) \mid \inf_{z \in (\delta, 1-\delta)} \|u - u^z\|_{L^2(\Omega)} < e^{-\delta/2\varepsilon}\}.$$

We will confine our analysis of the Allen–Cahn equation to the tubular neighborhood $\mathscr{N}(\delta)$ of $\mathscr{M}$.

By a straighforward adaptation of the proof of Lemma 2.4 in [3] (see also [9, 12] for similiar results) we obtain:

PROPOSITION 1.1 (Local Coordinates). *For each $u \in \mathscr{N}(\delta)$ there exists a unique $\xi \in (\delta, 1-\delta)$ such that*

$$\|u - u^{\xi}\|_{L^2(\Omega)} := \inf_{z \in (\delta, 1-\delta)} \|u - u^z\|_{L^2(\Omega)}.$$

Moreover ξ is a smooth function of u and if we set $v^{\xi} = u - u^{\xi}$ then $\int_{\Omega} v^{\xi} u^{\xi}_{\xi} = 0$, where $u^{\xi}_{\xi} = \partial u^{\xi}/\partial\xi$.

The main idea of this paper is to describe the flow determined by (1.1) near $\mathcal{M}$ in terms of the coordinates (ξ, v^ξ), ξ being the tangential and v^ξ the transversal direction to $\mathcal{M}$. For that we need to derive equations satisfied by (ξ, v^ξ). Replacing u by $u^\xi + v^\xi$ in (1.1) we obtain

$$\begin{cases} \Delta v^\xi - \varepsilon^{-2} F''(u^\xi)\, v^\xi - v^\xi_t = \xi_t u^\xi_\xi + N(v^\xi), & \text{in} \quad \Omega \times (0, T), \\ \dfrac{\partial v^\xi}{\partial n} = -\dfrac{\partial u^\xi}{\partial n}, & \text{on} \quad \partial\Omega \times (0, T), \\ v^\xi(x, 0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $N(v^\xi) = \varepsilon^{-2}[F'(u^\xi + v^\xi) - F'(u^\xi) - F''(u^\xi)\, v^\xi]$. Observe that in the above we have made use of (1.4). In what follows we refer to (1.6) as the v-equation.

Multiplying (1.6) by u^ξ_ξ and integrating by parts we get

$$\begin{aligned} &\xi_t[\, \|u^\xi_\xi\|^2_{L^2(\Omega)} - \langle u^\xi_{\xi\xi}, v^\xi \rangle] \\ &\qquad = -\int_{\partial\Omega} u^\xi_\xi \frac{\partial u^\xi}{\partial n}\, dS - \int_{\partial\Omega} v^\xi \frac{\partial u^\xi_\xi}{\partial n}\, dS - \langle N(v^\xi), u^\xi_\xi \rangle, \end{aligned} \tag{1.7}$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product; above we have made use of

$$\langle u^\xi_{\xi\xi} \xi_t, v^\xi \rangle = -\langle u^\xi_\xi, v^\xi_t \rangle,$$

and (1.4b). (1.7) will eventually provide the approximate formula for the speed of the interface ξ_t.

We can now state the main results of this paper.

Theorem 1.2 (Stability of $\mathcal{M}$). *Let $u(x, t)$, $x \in \Omega$, $t > 0$ be a solution to* (1.1) *and let $T_\delta = \sup\{t \mid u(\cdot, t) \in \mathcal{N}(\delta)\}$. Assume that $u(\cdot, 0) = u^\xi(\cdot, 0) + v^\xi(\cdot, 0)$ satisfies $\|v^\xi(\cdot, 0)\|_{C^0(\Omega)} \leqslant e^{-1/\varepsilon}$. The following statements hold true*

(i) *For $0 < t < T_\delta$ we have $u(\cdot, t) = u^\xi(\cdot, t) + v^\xi(\cdot, t)$, where $(\xi(t), v^\xi(\cdot, t))$ satisfy* (1.6), (1.7). *Moreover there exists a constant $\gamma > 0$ such that*

$$\|v^\xi(\cdot, t)\|_{C^0(\Omega)} \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}, \tag{1.8}$$

where we have set $\mathrm{d}^\xi := \min\{\xi, 1 - \xi\}$.

(ii) *If $T_\delta < \infty$ then $\mathrm{d}^{\xi(T_\delta)} = \delta$ and $T_\delta > |\xi(0) - \xi(T_\delta)|\, \varepsilon^\gamma e^{2\delta/\varepsilon}$.*

THEOREM 1.3 (The Law of Motion). *Let $u(\cdot, t)$, T_δ be as in the previous theorem and ξ satisfy* (1.7). *The following formula holds,*

$$\xi_t \, \|u^\xi_\xi\|^2_{L^2(\Omega)} \, (1+o(1)) = -\frac{d}{d\xi} \, j_\varepsilon(\xi) + r_\varepsilon(t), \tag{1.9}$$

where

$$j_\varepsilon(\xi) = \int_\Omega [\,|\nabla u^\xi|^2 + \varepsilon^{-2} F(u^\xi)] \, dx$$

$$\varepsilon^2 \, \frac{d}{d\xi} \, j_\varepsilon(\xi) = \beta^2 \left[K^- \left(\frac{\varepsilon}{2}\right)^{\alpha^- + 1} \Gamma(\alpha^- + 1) \, e^{-2\xi/\varepsilon} - K^+ \left(\frac{\varepsilon}{2}\right)^{\alpha^+ + 1} \Gamma(\alpha^+ + 1) \, e^{-2(1-\xi)/\varepsilon} \right] (1 + o(1)),$$

$$\varepsilon^2 \, |r_\varepsilon(t)| \leqslant C(\varepsilon^{2\alpha^- + 1} e^{-2\xi/\varepsilon} + \varepsilon^{2\alpha^+ + 1} e^{-2(1-\xi)/\varepsilon}),$$

$K^\pm$, $\alpha^\pm$ *are the constants in* (D4), Γ *denotes the standard gamma function and* β *is a constant depending on the nonlinearity* $F'(\cdot)$ *only.*

We will now outline the main points in the proofs of the above theorems. The analysis of the v-equation is based on the following spectral estimate.

PROPOSITION 1.4. *Let* $v \in L^2(\Omega)$ *be such that* $\int_\Omega v u^\xi_\xi \, dx = 0$ *for some* $\xi \in (\delta, 1-\delta)$. *We then have*

$$\int_\Omega [\,|\nabla v|^2 + \varepsilon^{-2} F''(u^\xi) \, v^2] \, dx \geqslant C\varepsilon^2 \, \|v\|^2_{H^1_\varepsilon(\Omega)}, \tag{1.10}$$

where $\|v\|^2_{H^1_\varepsilon(\Omega)} := \int_\Omega |\nabla v|^2 + \varepsilon^{-2} v^2$.

The proof of this proposition can be found in [6] for the case $N = 2$. The same argument, based on the so called Sola–Morales example (see [7]), applies when $N > 2$ with only minor changes. We omit the details here and refer the reader to [6].

Utilizing (1.10) we derive the following estimate

$$C \, \|v^\xi\|^2_{L^2(\Omega)} + \frac{d}{dt} \, \|v^\xi\|^2_{L^2(\Omega)} \leqslant \varepsilon^{-\gamma} \left\| \frac{\partial u^\xi}{\partial n} \right\|^2_{L^2(\partial\Omega)}. \tag{1.11}$$

Under the assumption $0 < t < T_\delta$ we also derive a preliminary estimate for ξ_t which basically states that we need to wait an exponentially long time for the interface to move an appreciable distance. Combining this observation, (1.11) and Gronwall's inequality we obtain estimate of the form

$$\|v^{\xi}(\cdot,t)\|_{L^2(\Omega)} \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}, \qquad 0<t<T_\delta. \tag{1.12}$$

This estimate followed by a bootstrap argument gives (1.8). The second assertion of Theorem 1.2 is then easy to conclude.

The proof of Theorem 1.3 is more delicate. It turns out that in justifying formula (1.9) further refinement of (1.8) is needed. This estimate, (so called improved v-estimate) has been known since the work of Fusco and Hale ([22]) and is in fact a key point in the approach we are following in this paper (see also [2]). Here we shall derive

$$\|v^{\xi}(\cdot,t)\|_{L^2(\Omega_{\delta/2})} \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \qquad c,\gamma>0, \quad 0<t<T_\delta, \tag{1.13}$$

where $\Omega_{\delta/2}=\{\delta/2<x_1<1-\delta/2\}$. To show (1.13) we first write the v-equation in the form

$$L^{\varepsilon,O} v^{\xi} = \Phi,$$

where $L^{\varepsilon,O}=\Delta-\partial/\partial t-\varepsilon^{-2}I$. We refer to $L^{\varepsilon,O}$ as the outer operator. We then utilize the single layer potential theory to solve "explicitly" the above equation. This provides us with estimates for v^{ξ} far from the transition layer. Finally by using again Proposition 1.4 we derive (1.13). The improved v-estimate in particular allows us to infer that the term involving the nonlinearity $N(v^{\xi})$ in (1.7) is negligible.

To demonstrate that the same is true for the boundary integral in (1.7) we need another refinement of (1.8). This time we have to analyze carefully the behaviour of v^{ξ} near the points $\{x_1=0\}\cap\partial\Omega$, $\{x_1=1\}\cap\partial\Omega$ ("corners" of Ω). Those points are the closest points to the interface such that $\partial u^{\xi}_{\xi}/\partial n\neq 0$. Here we employ again the single layer potential theory to derive the representation formula for v^{ξ} on $\partial\Omega$. Using then (1.13) and the decaying property of the fundamental solution of $L^{\varepsilon,O}$ we conclude the error estimate (1.7).

Equations (1.8), (1.9) provide very accurate description of the dynamics of (1.1) assuming that $|\xi-1/2|>C\varepsilon\,|\ln\varepsilon|$ with C sufficiently large so that j'_{ε} is truly the dominating term in the asymptotic expansion of ξ_t. It is natural to ask what happens when $|\xi-1/2|=O(\varepsilon\,|\ln\varepsilon|)$ and both terms in (1.9) become comparable. In [6] we gave a complete description of this situation for the case when Ω is two dimensional domain. We showed that if $K^+K^->0$ then there exists $\hat{\xi}$, $|\hat{\xi}-1/2|<C\varepsilon\,|\ln\varepsilon|$ such that there is an equilibrium of (1.1) near $\mathscr{M}$, $\hat{u}=u^{\hat{\xi}}+v^{\hat{\xi}}$, $u^{\hat{\xi}}\in\mathscr{M}$, $\int_{\Omega} u^{\hat{\xi}}_{\xi} v^{\hat{\xi}}\,dx=0$ and $\|v^{\hat{\xi}}\|_{L^2(\Omega)}\leqslant\varepsilon^{-\gamma}e^{\mathrm{d}^{\hat{\xi}}}$, $\gamma>0$. We also established that if $K^+K^-<0$ then there are no equilibria of (1.1) near $\mathcal{M}$. Finally the stability of the steady state solutions was discussed and we proved that the equilibria are stable if $K^{\pm} < 0$ and unstable if $K^{\pm} > 0$. We remark that the analysis in [6] can be generalized to $N > 2$, however we shall not pursue it here.

During the preparation of this paper we learned about some work on related issues by Ei and Yanagida [18]. They considered strip-like two dimensional domains in which the motion by mean curvature degenerates. However, in their case functions u^{ξ} do not satisfy the boundary conditions near the interface and as a result the normal velocity of the interface is of order $O(\varepsilon^2)$.

In this paper C, c, γ stand for generic positive constants whose value may change from line to line. We shall also denote $\langle u, v \rangle = \int_{\Omega} uv\, dx$, $\|v\|^2_{H^1_\varepsilon(\Omega)} = \int_{\Omega} [|\nabla v|^2 + \varepsilon^{-2} |v|^2]\, dx$, $\mathrm{d}^{\xi} := \min\{\xi, 1-\xi\}$.

This paper is organized as follows: in Section 2 we give the proof of Theorem 1.2. Section 3 contains the proof of Theorem 1.3. In Section 4 we establish a technical lemma which is used in Section 3.

2. STABILITY OF THE FLOW NEAR $\mathcal{M}$

2.1. *Proof of Theorem* 1.2—L^2 *estimates*

Let the functional J_ε be defined as

$$J_\varepsilon[u] = \int_{\Omega} [|\nabla u|^2 + \varepsilon^{-2} F(u)]\, dx.$$

By a straighforward calculation one finds that (1.1) is a gradient flow of J_ε in $L^2(\Omega)$. By a classical theory for the evolution equations (see for example [24]) one can show then that (1.1) has a global, unique, classical solution.

In this section we shall initially assume that $F(u)$ is an affine function for $|u| > 2$. Later we will see that the solutions of (1.1) we are interested in satisfy $|u| < 3/2$ (Lemma 2.8 to follow) and thus this extra hypothesis becomes irrelevant.

We first establish a technical lemma.

LEMMA 2.1. *Let $F(u)$ be as in* (1.1) *and assume that $F(u)$ is an affine function for $|u| > 2$. Let $v \in H^1_\varepsilon(\Omega)$ and $\sigma \in [0, 1]$ be such that $L^{2+\sigma}(\Omega) \hookrightarrow H^1_\varepsilon(\Omega)$. If $N(v) = \varepsilon^{-2}[F'(u^{\xi} + v) - F'(u^{\xi})\, v - F''(u^{\xi})\, v^2]$, $\xi \in (\delta, 1-\delta)$ then we have*

$$|\langle N(v), v \rangle| \leqslant C\varepsilon^{-2} \|v\|^2_{H^1_\varepsilon(\Omega)} \|v\|^{\bar{\sigma}}_{L^2(\Omega)}, \tag{2.1}$$

where $\bar{\sigma} = 2\sigma/(2+\sigma)$.

Proof. Since for $|v| < 4$, $|N(v)| < C\varepsilon^{-2} |v|^2$ and for $|v| \geqslant 4$, $N(v) \equiv 0$ therefore

$$\begin{aligned}|\langle N(v), v\rangle| &\leqslant \int_{\Omega \cap \{|v| < 4\}} |N(v)\, v|\, dx \\ &\leqslant C\varepsilon^{-2} \int_{\Omega \cap \{|v| < 4\}} |v|^{2+\sigma}\, dx \\ &\leqslant C\varepsilon^{-2} \|v\|^2_{H^1_\varepsilon(\Omega)} \left\{\int_{\Omega \cap \{|v| < 4\}} |v|^{2+\sigma}\, dx\right\}^{\sigma/(2+\sigma)} \\ &\leqslant C\varepsilon^{-2} \|v\|^2_{H^1_\varepsilon(\Omega)} \left\{\int_\Omega |v|^2\, dx\right\}^{\sigma/(2+\sigma)}.\end{aligned}$$

The proof is complete. ▮

In the next two lemmas we shall establish preliminary estimates for the coordinates (ξ, v^ξ). Although quite elementary these estimates play an important role throughout this paper.

LEMMA 2.2. *Let $u(x, t)$, $(x, t) \in \Omega \times (0, \infty)$ be a solution to* (1.1) *with $u(\cdot, 0) \in \mathcal{N}(\delta)$. If we set*

$$T_\delta = \sup\{t \mid u(\cdot, t) \in \mathcal{N}(\delta)\},$$

then there exist constants $C, \gamma > 0$ such that as long as $0 \leqslant t < T_\delta$

$$\begin{aligned}\|v^\xi(\cdot, t)\|^2_{L^2(\Omega)} &\leqslant \|v^\xi(x, 0)\|^2_{L^2(\Omega)}\, e^{-Ct} \\ &\quad + \varepsilon^{-\gamma} \int_0^t \left\| \frac{\partial u^\xi(\cdot, s)}{\partial n} \right\|^2_{L^2(\partial\Omega)} e^{-C(t-s)}\, ds,\end{aligned} \tag{2.2}$$

where v^ξ is a function satisfying (1.6).

Proof. From Proposition 1.1 we have $u(\cdot, t) = u^\xi(\cdot, t) + v^\xi(\cdot, t)$ as long as $t < T_\delta$, where v^ξ satisfies (1.6) and

$$\inf_{z \in (\delta, 1-\delta)} \|u - u^z\|_{L^2(\Omega)} = \|v^\xi(\cdot, t)\|_{L^2(\Omega)} \leqslant e^{-\delta/2\varepsilon}.$$

Multiplying (1.6) by v^ξ and integrating by parts we obtain

$$-B_\varepsilon(v^\xi, v^\xi) - \frac{1}{2}\frac{d}{dt} \|v^\xi\|^2_{L^2(\Omega)} = \langle N(v^\xi), v^\xi\rangle + \int_{\partial\Omega} v^\xi \frac{\partial u^\xi}{\partial n}\, dS, \tag{2.3}$$

where $B_\varepsilon(v, v) = \int_\Omega |\nabla v|^2 + \varepsilon^{-2} F''(u^\xi)\, v^2$. Let $0 < \sigma < 1$ be such that $L^{2+\sigma}(\Omega) \hookrightarrow H^1(\Omega)$. As long as $0 \leqslant t < T_\delta$ we have $\xi(t) \in (\delta, 1-\delta)$ hence applying Lemma 2.1 we find

$$|\langle N(v^\xi), v^\xi\rangle| \leqslant C\varepsilon^{-2}\, \|v^\xi\|^2_{H^1_\varepsilon(\Omega)}\, \|v^\xi\|^{\bar\sigma}_{L^2(\Omega)} \leqslant \varepsilon^3\, \|v^\xi\|^2_{H^1_\varepsilon(\Omega)},$$

provided that ε is taken small enough.

By the interpolation inequality (see (2.21) p. 69 in [27]) we have

$$\begin{aligned}
\left|\int_{\partial\Omega} v^\xi \frac{\partial u^\xi}{\partial n}\, dS\right| &\leqslant C\, \|v^\xi\|_{L^2(\partial\Omega)} \left\|\frac{\partial u^\xi}{\partial n}\right\|_{L^2(\partial\Omega)} \\
&\leqslant C\, \|v^\xi\|_{H^1(\Omega)} \left\|\frac{\partial u^\xi}{\partial n}\right\|_{L^2(\partial\Omega)} \\
&\leqslant \varepsilon^3\, \|v^\xi\|^2_{H^1_\varepsilon(\Omega)} + \varepsilon^{-\gamma} \left\|\frac{\partial u^\xi}{\partial n}\right\|^2_{L^2(\partial\Omega)}.
\end{aligned}$$

From Proposition 1.4 we also get

$$B_\varepsilon(v^\xi, v^\xi) \geqslant C\varepsilon^2\, \|v^\xi\|^2_{H^1_\varepsilon(\Omega)}.$$

Hence combining the last three inequalities and using (2.3) we obtain

$$C\, \|v^\xi\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\, \|v^\xi\|^2_{L^2(\Omega)} \leqslant \varepsilon^{-\gamma} \left\|\frac{\partial u^\xi}{\partial n}\right\|^2_{L^2(\partial\Omega)}.$$

The assertion of the lemma follows now by Gronwall's inequality. ∎

The following lemma quantifies the idea that the motion of the interface takes place at an exponential (in ε) rather than algebraic time scale.

LEMMA 2.3. *Let $u(\cdot, t)$, T_δ be as in the previous lemma. Let $k \geqslant 0$ be given and $0 \leqslant t_1 \leqslant t_2 < T_\delta$ be such that $|t_1 - t_2| < \varepsilon^{-k}$. If $\xi(t) \in (\delta, 1-\delta)$ satisfies* (1.7) *then for all sufficiently small ε we have*

$$e^{-\mathrm{d}^{\xi(t_1)}/\varepsilon} \leqslant 9 e^{-\mathrm{d}^{\xi(t_2)}/\varepsilon}. \tag{2.4}$$

Proof. We first show that if $u(\cdot, t) \in \mathcal{N}(\delta)$, then there exists $\gamma > 0$ such that for each sufficiently small ε we have

$$|\xi_t| \leqslant \varepsilon^{-\gamma} (e^{-\mathrm{d}^{\xi(t)}/\varepsilon} + \|v^\xi\|_{L^2(\Omega)})^2, \qquad 0 < t < T_\delta. \tag{2.5}$$

Observe that

$$\|u^\xi_\xi\|^2_{L^2(\Omega)} = O(\varepsilon^{-1}), \qquad |\langle v^\xi, u^\xi_{\xi\xi}\rangle| \leqslant \varepsilon^{-\gamma} e^{-\delta/2\varepsilon}.$$

Hence by (1.7)

$$|\xi_t| \leqslant C\varepsilon \left\{ \left| \int_{\partial\Omega} u_\xi^\xi \frac{\partial u^\xi}{\partial n} \, dS \right| + \int_\Omega |N(v^\xi) \, u_\xi^\xi| \, dx + \left| \int_{\partial\Omega} v^\xi \frac{\partial u_\xi^\xi}{\partial n} \, dS \right| \right\}$$
$$\leqslant C\varepsilon \{ I + II + III \}.$$

From (1.5) we find

$$I \leqslant \varepsilon^{-\gamma} e^{-2\mathrm{d}^\xi/\varepsilon}, \qquad \varepsilon \text{ sufficiently small.}$$

From $u(\cdot, t) \in \mathcal{N}(\delta)$, $0 \leqslant t < T_\delta$

$$II \leqslant \varepsilon^{-\gamma} \, \|v^\xi\|^2_{L^2(\Omega)}.$$

By applying first Hölder's inequality and then inequality (2.21) p. 69 in [27] we find

$$III \leqslant \|v^\xi\|_{L^{q_N}(\partial\Omega)} \left\| \frac{\partial u_\xi^\xi}{\partial n} \right\|_{L^{p_N}(\partial\Omega)} \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon} \, \|v^\xi\|_{L^2(\Omega)},$$

where $q_N = 2(N-1)/N$ and $p_N = q_N^*$. Combining the last three estimates we conclude (2.5).

By taking ε smaller if necessary we then get

$$|\xi(t_1) - \xi(t_2)| \leqslant \int_{t_1}^{t_2} |\xi_t| \, dt \leqslant \varepsilon^{-k} e^{-\delta/2\varepsilon} \leqslant 2\varepsilon.$$

The assertion of the lemma follows now from

$$\mathrm{d}^{\xi(t_1)} \geqslant \mathrm{d}^{\xi(t_2)} - |\xi(t_1) - \xi(t_2)|.$$

The proof is complete. ∎

We can now establish a "weaker" version of Theorem 1.2; the difference is that C^0 estimate (1.8) is replaced below by L^2 estimate.

LEMMA 2.4. *Let $u(x, t)$ and T_δ be as in Lemma 2.2. Assume that $u(\cdot, 0) = u^\xi(\cdot, 0) + v^\xi(\cdot, 0)$ is such that $\|v^\xi(\cdot, 0)\|_{L^2(\Omega)} \leqslant e^{-1/\varepsilon}$.*
The following statements hold true:

(1) *For $0 \leqslant t < T_\delta$ we have $u(\cdot, t) = u^\xi(\cdot, t) + v^\xi(\cdot, t)$ where $(\xi(t), v^\xi(\cdot, t))$ satisfy* (1.6) *and* (1.7). *Moreover*

$$\|v^\xi(\cdot, t)\|_{L^2(\Omega)} \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \tag{2.6}$$

(2) *If* $T_\delta < \infty$ *then we have* $\mathrm{d}^{\xi(T_\delta)} = \delta$. *In addition*

$$T_\delta > \varepsilon^\gamma e^{2\delta/\varepsilon}\, |\xi(0) - \xi(T_\delta)|. \tag{2.7}$$

Proof. (1) The first part of the assertion (1) follows directly from Proposition 1.1 and the definition of T_δ. We shall now prove estimate (2.6). From Lemma 2.2 and the hypothesis $\|v^\xi(\cdot, 0)\|_{L^2(\Omega)} \leqslant e^{-1/\varepsilon}$ we see that it sufficies to estimate

$$\begin{aligned}
&\int_0^t \left\| \frac{\partial u^\xi(\cdot, s)}{\partial n} \right\|^2_{L^2(\partial\Omega)} e^{-C(t-s)}\, ds \\
&\quad = \left\{ \int_0^{t-\varepsilon^{-2}} + \int_{t-\varepsilon^{-2}}^t \right\} \left\| \frac{\partial u^\xi(\cdot, s)}{\partial n} \right\|^2_{L^2(\partial\Omega)} e^{-C(t-s)}\, ds \\
&\quad \leqslant \varepsilon^{-\gamma} \left\{ \int_0^{t-\varepsilon^{-2}} e^{-\mathrm{d}^{\xi(s)}/\varepsilon} e^{-C/\varepsilon^2}\, ds + 9 e^{-\mathrm{d}^{\xi(t)}/\varepsilon} \int_{t-\varepsilon^{-2}}^t e^{-C(t-s)}\, ds \right\} \\
&\quad \leqslant \varepsilon^{-\gamma} (e^{-C/\varepsilon^2} + e^{-\mathrm{d}^{\xi(t)}/\varepsilon}) \\
&\quad \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}, \qquad 0 \leqslant t < T_\delta,
\end{aligned}$$

where we have made use of estimates (1.5) and Lemma 2.3. The proof of (1) is complete.

(2) Observe that from (2.6) is follows in particular that as long as $0 \leqslant t < T_\delta$

$$\inf_{z \in (\delta, 1-\delta)} \|u(\cdot, t) - u^z(\cdot, t)\|_{L^2(\Omega)} = \|v^\xi(\cdot, t)\|_{L^2(\Omega)} \leqslant e^{-3\delta/4\varepsilon},$$

and therefore the only way $u(\cdot, t)$ may "leave" $\mathcal{N}(\delta)$ is if $\xi(T_\delta) = \delta$ or $1 - \delta$ hence the first part of (2) follows.

To prove estimate (2.7) we utilize (2.6) in (2.5) to find

$$|\xi_t| \leqslant \varepsilon^{-\gamma} e^{-2\mathrm{d}^{\xi(t)}/\varepsilon}, \qquad 0 \leqslant t < T_\delta.$$

From

$$|\xi(0) - \xi(T_\delta)| \leqslant \int_0^{T_\delta} |\xi_t|\, dt,$$

we now easily conclude (2.7). The proof of the lemma is complete. ∎

2.2. *Bootstrap Argument*

In this subsection we shall improve estimate (2.6) from L^2 norm to C^0 norm.

We first need some preparation. Let $\Gamma(x, t; y, s)$, $(x, y) \in \Omega \times \Omega$, $t > s$ be the fundamental solution of

$$L^{\varepsilon, O} := \Delta - \frac{\partial}{\partial t} - \varepsilon^{-2} I \qquad \text{(the outer operator)}.$$

It is easy to see that $\Gamma(x, t; y, s) = e^{-(t-s)/\varepsilon^2} Z(x, t; y, s)$ where Z is the heat kernel

$$Z(x, t; y, s) = [4\pi(t-s)]^{-N/2} e^{-|x-y|^2/4(t-s)}.$$

In the sequel we shall often write (1.6) in the form

$$L^{\varepsilon, O} v^{\xi} = \Phi, \tag{2.8a}$$

where

$$\Phi := \xi_t u^{\xi}_{\xi} + N(v^{\xi}) + \varepsilon^{-2}[F''(u^{\xi}) - 1] \, v^{\xi}, \tag{2.8b}$$

and assume that v^{ξ} satisfies the following initial-boundary conditions

$$\begin{cases} \dfrac{\partial v^{\xi}}{\partial n} = -\dfrac{\partial u^{\xi}}{\partial n} & \text{on} \quad \partial\Omega \times (0, T_{\delta}) \\ v^{\xi}(x, 0) = v_0(x) & \text{where} \quad \|v_0\|_{C^0(\Omega)} \leqslant e^{-1/\varepsilon}. \end{cases} \tag{2.8c}$$

The key idea for the rest of this paper is to use the single layer potential to "solve" (2.8).

We shall now state, without proofs two simple technical lemmas.

LEMMA 2.5. *There exists a constant $C = C(\Omega)$ such that if $\partial\Omega \in C^{1, \alpha}$ then*

$$\sup_{x \in \partial\Omega} \int_{\partial\Omega} \left| \frac{\partial Z(x, t; y, s)}{\partial n_x} \right| dS_y \leqslant C(t-s)^{\alpha/2 - 1}, \qquad t > s,$$

where $Z(x, t; y, s)$ is the heat kernel.

LEMMA 2.6 (Gronwall inequality, cf. [24]). *Let $\psi(s) \geqslant 0$, $\Psi(s) \geqslant 0$, $0 \leqslant s \leqslant t$ be continuous functions satisfying*

$$\psi(t) \leqslant K \int_0^t \psi(s) \, e^{-(t-s)/\varepsilon^2} (t-s)^p \, ds + \Psi(t), \qquad p > -1, \quad K > 0.$$

Then

$$\psi(t) \leqslant \Psi(t) + \int_0^t \Psi(s)\, E_{\varepsilon,\,p}(t-s)\, ds,$$

where

$$E_{\varepsilon,\,p}(\tau) = e^{-\tau/\varepsilon^2} \sum_{i=1}^{\infty} \frac{\tau^{p_i} \Gamma^i(1+p)\, K^i}{\Gamma(p_i - 1)}, \qquad \tau > 0, \quad p_i = (1+p)\, i - 1,$$

and Γ is the standard gamma function. Moreover, there exists a positive constant C depending on p, K only such that

$$E_{\varepsilon,\,p}(\tau) \leqslant C e^{-\tau/2\varepsilon^2} \tau^p.$$

After this preparation we can set up our bootstrap argument. This will be done in the next two lemmas.

LEMMA 2.7. *Let $z^{\xi}(x, t)$, $(x, t) \in \Omega \times (0, T_\delta)$ be a solution of*

$$\begin{cases} L^{\varepsilon,\,O} z^{\xi} = 0 & \text{in} \quad \Omega \times (0, T_\delta), \\ \dfrac{\partial z^{\xi}}{\partial n} = -\dfrac{\partial u^{\xi}}{\partial n} & \text{on} \quad \partial\Omega \times (0, T_\delta), \\ z^{\xi}(x, 0) = v^{\xi}(x, 0) & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{2.9}$$

where $\xi = \xi(t) \in (\delta, 1-\delta)$. The following estimate holds

$$\|z^{\xi}(\cdot, t)\|_{C^0(\Omega)} \leqslant C(e^{-\mathrm{d}^{\xi(t)}/\varepsilon} + \|v^{\xi}(\cdot, 0)\|_{C^0(\Omega)}). \tag{2.10}$$

Proof. The solution of (2.9) can be written explicitly in the form

$$z^{\xi}(x, t) = \int_0^t \int_{\partial\Omega} \Gamma(x, t; y, s)\, \phi(y, s)\, dS_y\, ds$$
$$+ \int_{\Omega} \Gamma(x, t; y, 0)\, v^{\xi}(y, 0)\, dy, \tag{2.11a}$$

where

$$\frac{1}{2}\,\phi(x, t) = \int_0^t \int_{\partial\Omega} \frac{\partial \Gamma(x, t; y, s)}{\partial n_x}\, \phi(y, s)\, dS_y\, ds + \frac{\partial u^{\xi}(x, t)}{\partial n}$$
$$+ \int_{\Omega} \frac{\partial \Gamma(x, t; y, 0)}{\partial n_x}\, v^{\xi}(y, 0)\, dy. \tag{2.11b}$$

We first estimate $\|\phi(\cdot, t)\|_{C^0(\partial\Omega)}$, $0<t<T_\delta$. From (2.11b) and Lemma 2.5 we obtain by direct calculation

$$\begin{aligned}\frac{1}{2}\|\phi(\cdot, t)\|_{C^0(\partial\Omega)} &\leqslant \int_0^t \int_{\partial\Omega} \left|\frac{\partial\Gamma(x, t; y, s)}{\partial n_x}\right| \|\phi(\cdot, s)\|_{C^0(\partial\Omega)}\, dS_y\, ds \\ &\quad + \left\|\frac{\partial u^\xi(\cdot, t)}{\partial n}\right\|_{C^0(\partial\Omega)} + \int_\Omega \left|\frac{\partial\Gamma(x, t; y, 0)}{\partial n_x}\right| \|v^\xi(\cdot, 0)\|_{C^0(\Omega)}\, dy \\ &\leqslant C\int_0^t \|\phi(\cdot, s)\|_{C^0(\partial\Omega)}\, e^{-(t-s)/\varepsilon^2}(t-s)^{\alpha/2-1}\, ds \\ &\quad + \left\|\frac{\partial u^\xi(\cdot, t)}{\partial n}\right\|_{C^0(\partial\Omega)} + t^{-1/2}e^{-t/\varepsilon^2}\, \|v^\xi(\cdot, 0)\|_{C^0(\Omega)}.\end{aligned}$$

Applying now Lemma 2.6 yields

$$\begin{aligned}\|\phi(\cdot, t)\|_{C^0(\partial\Omega)} \leqslant{}& \left\|\frac{\partial u^\xi(\cdot, t)}{\partial n}\right\|_{C^0(\partial\Omega)} + t^{-1/2}e^{-t/\varepsilon^2}\, \|v^\xi(\cdot, 0)\|_{C^0(\Omega)} \\ &+ \int_0^t E_{\varepsilon,\, \alpha/2-1}(t-s)\left[\left\|\frac{\partial u^\xi(\cdot, s)}{\partial n}\right\|_{C^0(\partial\Omega)}\right. \\ &\left. + s^{-1/2}e^{-s/\varepsilon^2}\, \|v^\xi(\cdot, 0)\|_{C^0(\Omega)}\right] ds. \qquad (2.12a)\end{aligned}$$

From (2.4) and (1.5) we get

$$\begin{aligned}&\int_0^t E_{\varepsilon,\, \alpha/2-1}(t-s)\left\|\frac{\partial u^\xi(\cdot, s)}{\partial n}\right\|_{C^0(\partial\Omega)} ds \\ &\leqslant C\varepsilon^{-1}\int_0^t e^{-(t-s)/2\varepsilon^2}(t-s)^{\alpha/2-1}e^{-\mathrm{d}^{\xi(s)}/\varepsilon}\, ds \\ &= C\varepsilon^{-1}\left\{\int_0^{t-6\varepsilon} + \int_{t-6\varepsilon}^t\right\} e^{-(t-s)/2\varepsilon^2}(t-s)^{\alpha/2-1}e^{-\mathrm{d}^{\xi(s)}/\varepsilon}\, ds \\ &\leqslant C\varepsilon^{-1}\left\{e^{-\mathrm{d}^{\xi(t)}/\varepsilon}\int_{t-6\varepsilon}^t e^{-(t-s)/\varepsilon^2}(t-s)^{\alpha/2-1}\, ds + e^{-1/\varepsilon}\right\} \\ &\leqslant C\varepsilon^{-1+\alpha}e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \qquad (2.12b)\end{aligned}$$

Similiar calculation yields

$$\int_0^t E_{\varepsilon,\,\alpha/2-1}(t-s)\ \|v^{\xi}(\cdot, 0)\|_{C^0(\Omega)}\,ds$$
$$\leqslant Ct^{-1/2+\alpha/2}e^{-t/\varepsilon^2}\ \|v^{\xi}(\cdot, 0)\|_{C^0(\Omega)}, \tag{2.12c}$$

and thus combining (2.12a, b, c) we conclude

$$\|\phi(\cdot, t)\|_{C^0(\partial\Omega)} \leqslant C(\varepsilon^{-1}e^{-\mathrm{d}^{\xi(t)}/\varepsilon}+t^{-1/2}e^{-t/\varepsilon^2}\ \|v^{\xi}(\cdot, 0)\|_{C^0(\Omega)}).$$

Taking C^0 norm on both sides of (2.11a), utilizing the above estimate and

$$\int_{\partial\Omega} |\Gamma(x, t; y, s)|\ dS_y \leqslant Ce^{-(t-s)/\varepsilon^2}(t-s)^{-1/2},$$

$$\int_{\Omega} |\Gamma(x, t; y, 0)|\ dy \leqslant C,$$

one obtains (2.10). ∎

Lemma 2.8. *Let $w^{\xi}(x, t)$, $(x, t)\in\Omega\times(0, T_\delta)$ be a solution of*

$$\begin{cases} L^{\varepsilon, O}w^{\xi}=\Phi & \text{in} \quad \Omega\times(0, T_\delta), \\ \dfrac{\partial w^{\xi}}{\partial n}=0 & \text{on} \quad \partial\Omega\times(0, T_\delta), \\ w^{\xi}(x, 0)=0 & \text{in} \quad \Omega\times\{0\}, \end{cases} \tag{2.13}$$

where Φ is defined in (2.8b). *There exists $\gamma>0$ such that for $t\in(0, T_\delta)$ we have*

$$\|w^{\xi}(\cdot, t)\|_{C^0(\Omega)} \leqslant \varepsilon^{-\gamma}e^{-\mathrm{d}^{\xi(t)}/\varepsilon}, \tag{2.14a}$$
$$\|v^{\xi}(\cdot, t)\|_{C^0(\Omega)} \leqslant \varepsilon^{-\gamma}e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \tag{2.14b}$$

Proof. Observe that once (2.14a) is established then (2.14b) follows immediately from $v^{\xi}=z^{\xi}+w^{\xi}$ and Lemma 2.7.

We will show (2.14a) by setting up an iteration scheme. Let $A_\varepsilon=-\Delta+\varepsilon^{-2}I$ with Neuman boundary condition. By the variation of constants formula we have

$$w^{\xi}(x, t)=\int_0^t e^{-A_\varepsilon(t-s)}\,\Phi\,ds. \tag{2.15}$$

Observe that A_ε is a positive, sectorial operator in $X=L^2(\Omega)$ with $\mathscr{D}(A_\varepsilon)\subset W^{2,2}(\Omega)$. Applying $A_\varepsilon^{1/2}$ to both sides of (2.15) and then taking L^2 norms we obtain, by the well known estimate (see for example Thm. 1.4.3 p. 26 in [24])

$$\|A_\varepsilon^{1/2} w^\xi(\cdot, t)\|_{L^2(\Omega)} \leqslant \int_0^t \|A_\varepsilon^{1/2} e^{-A_\varepsilon(t-s)}\| \; \|\Phi(\cdot, s)\|_{L^2(\Omega)}\, ds$$

$$\leqslant C \int_0^t (t-s)^{-1/2}\, e^{-(t-s)/\varepsilon^2}\, \|\Phi(\cdot, s)\|_{L^2(\Omega)}\, ds. \qquad (2.16)$$

We shall now estimate $\|\Phi(\cdot, s)\|_{L^2(\Omega)}$, $0 < s < T_\delta$. From (2.8b) we have

$$\|\Phi(\cdot, s)\|_{L^2(\Omega)} \leqslant |\xi_t| \; \|u^\xi_\xi\|_{L^2(\Omega)} + C\varepsilon^{-2} \|v^\xi\|_{L^2(\Omega)} + \|N(v^\xi)\|_{L^2(\Omega)}.$$

From (2.5), (2.6)

$$|\xi_t| \leqslant e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}.$$

Since $N(v^\xi) \leqslant C\,|v^\xi|^2$ for $|v^\xi| < 4$, $N(v^\xi) \equiv 0$ for $|v^\xi| \geqslant 4$ therefore by Lemma 2.4 we get

$$C\varepsilon^{-2} \|v^\xi(\cdot, t)\|_{L^2(\Omega)} + \|N(v^\xi(\cdot, t))\|_{L^2(\Omega)} \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon},$$

and thus by (2.16) and Lemma 2.3 we obtain

$$\|A_\varepsilon^{1/2} w^\xi(\cdot, t)\|_{L^2(\Omega)} \leqslant C\varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \qquad (2.17a)$$

From the imbedding $W^{1, q_1}(\Omega) \hookrightarrow X^{1/2}$, $q_1 < 2N/(2+N)$ it follows

$$\|w^\xi(\cdot, t)\|_{W^{1, q_1}(\Omega)} \leqslant C\varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \qquad (2.17b)$$

If $q_1 > N$ then (2.14a) follows from the imbedding $C^0(\Omega) \hookrightarrow W^{1, q_1}(\Omega)$, otherwise form Lemma 2.7 and the imbedding $L^{p_1}(\Omega) \hookrightarrow W^{1, q_1}(\Omega)$ we conclude

$$\|v^\xi(\cdot, t)\|_{L^{p_1}(\Omega)} \leqslant C\varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}, \qquad (2.17c)$$

where $p_1 = Nq_1/(N - q_1) > 2$. We can now repeat the whole argument above taking L^{p_1} instead of L^2 norms in (2.16). Consequently we improve estimates (2.17a, b, c) and in particular obtain an analog of (2.17b) with $q_2 > q_1$. After finite number of steps we will get $q_k > N$ and thus conclude (2.14a). The proof is complete. ▮

Proof of Theorem 1.2. Clearly (1.8) follows from Lemma 2.8 while (ii) is a direct consequence of Lemma 2.4. The proof is complete. ▮

Observe that in the view of (2.14b) the assumption that $F(u)$ is an affine function for $|u| > 4$ can now be dropped.

3. THE FORMULA FOR THE SPEED

3.1. *Local* (*in Space*) *Estimates*

In order to utilize the equation (1.7) to determine the asymptotic formula for the speed of the interface ξ_t we need much more detailed information about the transversal coordinate v^ξ than the one given in Theorem 1.2. In what follows we will derive local estimates for v^ξ: far from the interface (outer estimate), near the interface (inner estimate, here refered to as the improved v-estimate), on the boundary of Ω (boundary layer estimate). Those estimates are in some sense analogs of the outer, inner and boundary layer expansions of the classical method of formal asymptotic expansions.

We begin this section by stating an "outer" estimate for v^ξ. The proof of the lemma below requires some standard but technical arguments involving parabolic potentials. The details are given in Section 4.

LEMMA 3.1. *Let* $u(\cdot, t) \in \mathcal{N}(\delta)$ *be a solution to* (1.1) *with* $u(\cdot, 0) = u^\xi(\cdot, 0) + v^\xi(\cdot, 0)$, $\|v^\xi(\cdot, 0)\|_{C^0(\Omega)} \leqslant e^{-1/\varepsilon}$. *There exists a constant* $c > 0$ *depending on* δ *only such that for* $x \in \{x_1 \in (3\delta/8,\ 5\delta/8) \cup (1 - 5\delta/8,\ 1 - 3\delta/8)\} \cap \Omega$ *we have*

$$|v^\xi(x, t)| \leqslant e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \tag{3.1}$$

where $u(\cdot, t) = u^\xi(\cdot, t) + v^\xi(\cdot, t)$ *and* v^ξ *is a solution to* (1.6).

As we have mentioned already the Lemma below is the crucial point in the approach we are following here. With its help we can effectively control terms involving v^ξ in the equation (1.7). The proof given in this paper is the adaptation of a similiar result for the elliptic analog of (1.1) (see [6, 26]).

LEMMA 3.2 (Improved v-estimate). *Let* $u(\cdot, t)$ *be as in Lemma* 3.1. *There exists constant* $c > 0$ *depending on* δ *only such that for all suficiently small* ε *we have*

$$\|v^\xi(\cdot, t)\|_{L^2(\Omega_{\delta/2})} \leqslant e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \tag{3.2}$$

where $\Omega_{\delta/2} = \{x_1 \in (\delta/2, 1 - \delta/2)\} \cap \Omega$.

Proof. Multiplying (1.6) by v^ξ and integrating over $\Omega_{\delta/2}$ yields

$$\begin{aligned} -\int_{\Omega_{\delta/2}} [|\nabla v^\xi|^2 + \varepsilon^{-2} F''(u^\xi)(v^\xi)^2] - \frac{1}{2}\frac{d}{dt}\|v^\xi\|^2_{L^2(\Omega_{\delta/2})} \\ = \int_{\Omega_{\delta/2}} \xi_t u^\xi_\xi v^\xi + \int_{\Omega_{\delta/2}} N(v^\xi)\, v^\xi - \int_{\partial\Omega_{\delta/2}} \frac{\partial v^\xi}{\partial n} v^\xi\, dS \\ = I + II + III. \end{aligned} \tag{3.3}$$

It follows from (2.5) and (2.14b)

$$I \leqslant e^{-2\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \qquad c > 0.$$

From $|N(v^{\xi})| \leqslant C\,|v^{\xi}|^2$ we get

$$II \leqslant C\,\|v^{\xi}\|_{C^0(\Omega)}\,\|v^{\xi}\|^2_{L^2(\Omega_{\delta/2})} \leqslant e^{-c/\varepsilon}\,\|v^{\xi}\|^2_{L^2(\Omega_{\delta/2})}, \qquad c > 0.$$

By Lemma 3.1 and inequality (2.21) p. 69 in [27] we obtain

$$\begin{aligned} III &\leqslant \left\{\int_{\partial\Omega_{\delta/2} \cap \{x_1 = \delta/2,\, x_1 = 1-\delta/2\}} |v^{\xi}|^{p_N}\, dS\right\}^{1/p_N} \left\{\int_{\partial\Omega_{\delta/2}} \left|\frac{\partial v^{\xi}}{\partial n}\right|^{q_N} dS\right\}^{1/q_N} \\ &\leqslant e^{-\mathrm{d}^{\xi}/\varepsilon} e^{-c/\varepsilon} \left\{\int_{\partial\Omega_{\delta/2}} \left|\frac{\partial v^{\xi}}{\partial n}\right|^{q_N} dS\right\}^{1/q_N} \\ &\leqslant e^{-\mathrm{d}^{\xi}/\varepsilon} e^{-c/\varepsilon}\,\|v^{\xi}\|_{H^1(\Omega_{\delta/2})} \\ &\leqslant e^{-2\mathrm{d}^{\xi}/\varepsilon} e^{-c/\varepsilon} + \varepsilon^3\,\|v^{\xi}\|^2_{H^1_\varepsilon(\Omega_{\delta/2})}, \end{aligned}$$

where $q_N = 2(N-1)/N$ and $p_N = q_N^*$.

We shall now estimate the left hand side of (3.3). Observe that from $\langle u^{\xi}_{\xi}, v^{\xi}\rangle = 0$ we get

$$\left|\int_{\Omega_{\delta/2}} u^{\xi}_{\xi} v^{\xi}\, dx\right| \leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-\delta/2\varepsilon}. \tag{3.4}$$

Let $v^{\xi} = \bar{v}^{\xi} + \tilde{v}^{\xi}$, $\tilde{v}^{\xi} = u^{\xi}_{\xi} \int_{\Omega_{\delta/2}} u^{\xi}_{\xi} v^{\xi}$, $\int_{\Omega_{\delta/2}} \bar{v}^{\xi} u^{\xi}_{\xi} = 0$. We then have

$$\begin{aligned} &\int_{\Omega_{\delta/2}} |\nabla v^{\xi}|^2 + \varepsilon^{-2} F''(u^{\xi})(v^{\xi})^2\, dx \\ &\quad = \int_{\Omega_{\delta/2}} |\nabla \bar{v}^{\xi}|^2 + \varepsilon^{-2} F''(u^{\xi})(\bar{v}^{\xi})^2\, dx \\ &\qquad + \int_{\Omega_{\delta/2}} u^{\xi}_{\xi} v^{\xi}\, dx \left\{\int_{\Omega_{\delta/2}} u^{\xi}_{\xi} v^{\xi}\, dx \int_{\partial\Omega_{\delta/2}} u^{\xi}_{\xi}\,\frac{\partial u^{\xi}_{\xi}}{\partial n}\, dS + 2\int_{\partial\Omega_{\delta/2}} \bar{v}^{\xi}\,\frac{\partial u^{\xi}_{\xi}}{\partial n}\, dS\right\}. \end{aligned}$$

By the analog of the spectral estimate in Proposition 1.4 with Ω replaced by $\Omega_{\delta/2}$ and (3.4) we obtain

$$\begin{aligned} \int_{\Omega_{\delta/2}} |\nabla \bar{v}^{\xi}|^2 + \varepsilon^{-2} F''(u^{\xi})(\bar{v}^{\xi})^2\, dx &\geqslant C\,\|\bar{v}^{\xi}\|^2_{H^1_\varepsilon(\Omega_{\delta/2})} \\ &\geqslant C(\|v^{\xi}\|^2_{H^1_\varepsilon(\Omega_{\delta/2})} - \|\tilde{v}^{\xi}\|^2_{H^1_\varepsilon(\Omega_{\delta/2})}) \\ &\geqslant C\,\|v^{\xi}\|^2_{H^1_\varepsilon(\Omega_{\delta/2})} - e^{-2\mathrm{d}^{\xi}/\varepsilon} e^{-c/\varepsilon}. \end{aligned}$$

Furthermore using (3.4) again we find

$$\left|\int_{\partial\Omega_{\delta/2}} \bar{v}^{\xi} \frac{\partial u_{\xi}^{\xi}}{\partial n}\, dS\right| \leqslant \Big(\max_{x_1=\delta/2,\, x_1=(1-\delta)/2} \{|v^{\xi}(x)|\} + e^{-\mathrm{d}^{\xi}/\varepsilon} e^{-\delta/\varepsilon}\Big) \int_{\partial\Omega_{\delta/2}} \left|\frac{\partial u_{\xi}^{\xi}}{\partial n}\right| dS$$
$$\leqslant \varepsilon^{-\gamma} e^{-2\mathrm{d}^{\xi}/\varepsilon} e^{-c/\varepsilon},$$

hence by (3.4)

$$\int_{\Omega_{\delta/2}} [|\nabla v^{\xi}|^2 + \varepsilon^{-2} F''(u^{\xi})(v^{\xi})^2]$$
$$\geqslant C\varepsilon^2 \|v^{\xi}\|^2_{H^1_{\varepsilon}(\Omega_{\delta/2})} + O(e^{-2\mathrm{d}^{\xi}/\varepsilon} e^{-c/\varepsilon}), \qquad c>0. \tag{3.5}$$

Combining (3.3), estimates on I, II, III and (3.5) yields

$$C \|v^{\xi}(\cdot, t)\|^2_{L^2(\Omega_{\delta/2})} + \frac{1}{2}\frac{d}{dt} \|v^{\xi}(\cdot, t)\|^2_{L^2(\Omega_{\delta/2})} \leqslant e^{-2\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \qquad c>0,$$

and thus from Gronwall's inequality we conclude

$$\|v^{\xi}(\cdot, t)\|^2_{L^2(\Omega_{\delta/2})} \leqslant \|v^{\xi}(\cdot, 0)\|^2_{L^2(\Omega_{\delta/2})} + e^{-c/\varepsilon} \int_0^t e^{-2\mathrm{d}^{\xi(s)}/\varepsilon} e^{-C(t-s)/\varepsilon^2}\, ds.$$

Arguing as in Lemma 2.7 to handle the integral in the above expression we get (3.2). The proof is complete. ∎

Before proving the boundary estimate we need a technical lemma.

LEMMA 3.4. *Let v^{ξ} be a solution to* (1.6). *Then for each* $(x, t) \in \partial\Omega \times (0, T_{\delta})$ *we have*

$$-\frac{1}{2} v^{\xi}(x, t) = \int_0^t \int_{\partial\Omega} \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} v^{\xi}(y, s)\, dS_y\, ds$$
$$+ \int_0^t \int_{\partial\Omega} \frac{\partial u^{\xi}}{\partial n}(y, s)\, \Gamma(x, t; y, s)\, dS_y\, ds$$
$$+ \int_0^t \int_{\Omega} \Gamma(x, t; y, s)\, \Phi(y, s)\, dy\, ds - v_0(x), \tag{3.6}$$

where Φ is defined in (2.8b) *and Γ is the fundamental solution of $L^{\varepsilon, O}$.*

Proof. Let $x \in \partial\Omega$ be fixed and let $\{x_n\}_{n=1}^{\infty} \subset \Omega$ be such that $x_n \to x$ nontangentially. Multiplying (2.8a) by $\Gamma(x_n, t; y, s)$ and integrating by parts we get

$$-v^{\xi}(x_n, t) - \int_0^t \int_{\partial\Omega} v^{\xi}(y, s) \frac{\partial \Gamma(x_n, t; y, s)}{\partial n_y} \, dS_y \, ds$$

$$= \int_0^t \int_{\partial\Omega} \frac{\partial u^{\xi}}{\partial n}(y, s) \, \Gamma(x_n, t; y, s) \, dS_y \, ds$$

$$+ \int_0^t \int_{\Omega} \Phi(y, s) \, \Gamma(x_n, t; y, s) \, dy \, ds - v_0(x_n).$$

From the jump relation for the single layer potentials (see [21]) we obtain

$$\lim_{n \to \infty} \int_0^t \int_{\partial\Omega} v^{\xi}(y, s) \frac{\partial \Gamma(x_n, t; y, s)}{\partial n_y} \, dS_y \, ds$$

$$= -\frac{1}{2} v^{\xi}(x, t) + \int_0^t \int_{\partial\Omega} v^{\xi}(y, s) \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \, dS_y \, ds.$$

This yields (3.6). The proof is complete. ∎

LEMMA 3.4 (Boundary estimate). *There exists a constant $c > 0$ depending on δ such that*

$$|v^{\xi}(x, t)| \leqslant C(e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon} + \varepsilon^{\alpha^-} e^{-\xi(t)/\varepsilon}),$$

$$\text{for} \quad x \in \{x_1 < \delta/2\} \cap \partial\Omega := \partial\Omega^-, \tag{3.7a}$$

$$|v^{\xi}(x, t)| \leqslant C(e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon} + \varepsilon^{\alpha^+} e^{-(1-\xi(t))/\varepsilon}),$$

$$\text{for} \quad x \in \{x_1 > 1 - \delta/2\} \cap \partial\Omega := \partial\Omega^+. \tag{3.7b}$$

Proof (*cf.* [6]). The idea of the proof is similiar to the one in Lemma 2.7. Using the representation formula (3.6), applying Lemma 2.8 and utilizing the decay properties of the potential Γ we can establish that if $x \in \{x_1 < \delta/2\} \cap \partial\Omega$ then

$$-\frac{1}{2} v^{\xi}(x, t) = \int_0^t \int_{\partial\Omega} v^{\xi}(y, s) \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \, dS_y \, ds + \tilde{\Phi}(t),$$

where

$$|\tilde{\Phi}(t)| \leqslant C(e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon} + \varepsilon^{\alpha^-} e^{-\xi(t)/\varepsilon}), \qquad c > 0.$$

Taking x such that $|v^\xi(x,t)| = \sup_{y\in\partial\Omega^-} |v^\xi(y,t)|$ and using Lemma 3.1, Lemma 2.8 and the decay properties of the fundamental solution Γ we find

$$\begin{aligned}
&\left| \int_0^t \int_{\partial\Omega} v^\xi(y,s) \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \, dS_y \, ds \right| \\
&\quad \leqslant \int_0^t \left(\int_{\partial\Omega^-} + \int_{\partial\Omega \setminus \partial\Omega^-} \right) |v^\xi(y,s)| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dS_y \, ds \\
&\quad \leqslant \int_0^t \int_{\partial\Omega^-} |v^\xi(x,s)| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dS_y \, ds \\
&\qquad + \int_0^t \int_{\partial\Omega\setminus\partial\Omega^-} |v^\xi(y,s)| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dS_y \, ds \\
&\quad \leqslant \int_0^t \int_{\partial\Omega} |v^\xi(x,s)| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dS_y \, ds + e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \qquad c > 0,
\end{aligned}$$

hence it follows

$$\begin{aligned}
|v^\xi(x,t)| \leqslant 2 \int_0^t |v^\xi(x,t)| \int_{\partial\Omega} \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dS_y \, ds \\
+ C(e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon} + \varepsilon^{\alpha^-} e^{-\xi(t)/\varepsilon}). \qquad (3.8)
\end{aligned}$$

From (3.8) we obtain (3.7a) by applying Gronwall's inequality (Lemma 2.6). Estimate (3.7b) can be established by a similiar argument. The proof of the lemma is complete. ∎

3.2. *The Reduced Dynamics*

We set

$$j_\varepsilon(\xi) := J_\varepsilon[u^\xi], \qquad \delta < \xi < 1 - \delta.$$

We call $j_\varepsilon(\xi)$ the reduced energy.

LEMMA 3.5. (1) *Let* $\xi \in (\delta, 1-\delta)$. *We have*

$$\begin{aligned}
&\xi_t(\|u^\xi_\xi\|^2_{L^2(\Omega)} - \langle u^\xi_{\xi\xi}, v^\xi \rangle) \\
&\quad = -\frac{d}{d\xi} j_\varepsilon(\xi) - \int_{\partial\Omega} v^\xi \frac{\partial u^\xi_\xi}{\partial n} dS - \langle N(v^\xi), u^\xi_\xi \rangle. \qquad (3.9)
\end{aligned}$$

(2) *Let* $K^{\pm}$, $\alpha^{\pm}$ *be as in* (D4). *The following formula holds,*

$$\varepsilon^2 \frac{d}{d\xi} j_\varepsilon(\xi) = \beta^2 \left[K^- \left(\frac{\varepsilon}{2}\right)^{\alpha^- + 1} \Gamma(\alpha^- + 1)\, e^{-2\xi/\varepsilon} \right.$$

$$\left. - K^+ \left(\frac{\varepsilon}{2}\right)^{\alpha^+ + 1} \Gamma(\alpha^- + 1)\, e^{-2(1-\xi)/\varepsilon} \right] [1 + o(1)], \quad (3.10)$$

where Γ *is the standard gamma function and* β *depends on* $F'(\cdot)$ *only.*

Proof. (1) follows from the definition of j_ε and (1.4) after straightforward calculations.

(2) Let $D_a^- = \{x \in \partial\Omega \mid -a < x_1 < 0\}$, $D_a^+ = \{x \in \partial\Omega \mid 1 < x_1 < 1 + a\}$, where a is as in (D3). Then from (1.5) we get

$$\frac{d}{d\xi} j_\varepsilon(\xi) = \int_{D_a^-} u_\xi^\xi \frac{\partial u^\xi}{\partial n}\, dS + \int_{D_a^+} u_\xi^\xi \frac{\partial u^\xi}{\partial n}\, dS$$

$$+ O(\varepsilon^{-2} e^{-2(\xi + a)/\varepsilon} + \varepsilon^{-2} e^{-2(1 - \xi + a)/\varepsilon}). \quad (3.11)$$

In the calculation below we shall use the well known estimate

$$U'(\eta) = \beta e^{-|\eta|}(1 + O(e^{-|\eta|/2})), \qquad |\eta| \to \infty.$$

For the first integral above we can write

$$\int_{D_a^-} u_\xi^\xi \frac{\partial u^\xi}{\partial n}\, dS = -\varepsilon^{-2} \int_{D_a^-} \left[U'\left(\frac{x_1 - \xi}{\varepsilon}\right)\right]^2 n_1(x)\, dS_x$$

$$= -\varepsilon^{-2} \int_{-a}^{0} \int_{S^{N-2}} \left[U'\left(\frac{x_1 - \xi}{\varepsilon}\right)\right]^2 n_1(x_1, \theta) \left|\frac{\partial y^-}{\partial(x_1, \theta)}\right| d\theta\, dx_1$$

$$= -\varepsilon^{-2}\beta^2 \int_{-a}^{0} e^{-2(\xi - x_1)/\varepsilon} \phi^-(x_1)[1 + o(1)]\, dx_1$$

$$= \varepsilon^{-2}\beta^2 K^- \left(\frac{\varepsilon}{2}\right)^{\alpha^- + 1} \Gamma(\alpha^- + 1)\, e^{-2\xi/\varepsilon}(1 + o(1)).$$

Likewise

$$\int_{D_a^+} u_\xi^\xi \frac{\partial u^\xi}{\partial n}\, dS = -\varepsilon^{-2}\beta^2 K^+ \left(\frac{\varepsilon}{2}\right)^{\alpha^+ + 1} \Gamma(\alpha^+ + 1)\, e^{-2(1-\xi)/\varepsilon}(1 + o(1)).$$

We conclude now (3.10). The proof is complete. ∎

Proof of Theorem 1.3. From Lemma 3.5 we see that it sufficies to show that the following asymptotic formula holds

$$\xi_t \, \|u_\xi^\xi\|^2_{L^2(\Omega)} \, (1+o(1)) = -\frac{d}{d\xi} \, j_\varepsilon(\xi) + r_\varepsilon(t), \tag{3.12}$$

where

$$\varepsilon^2 \, |r_\varepsilon(t)| \leqslant C(\varepsilon^{2\alpha^- + 1} e^{-2\xi/\varepsilon} + \varepsilon^{2\alpha^+ + 1} e^{-2(1-\xi)/\varepsilon}). \tag{3.13}$$

Formula (3.12) follows immediately from (3.9). It remains to show (3.13). From Lemma 2.8, Lemma 3.2, (1.5) and $\delta < \xi < 1 - \delta$ we get

$$\begin{aligned}\left| \int_\Omega N(v^\xi) \, u_\xi^\xi \, dx \right| &\leqslant \left| \int_{\Omega_{\delta/2}} N(v^\xi) \, u_\xi^\xi \, dx \right| + \left| \int_{\Omega \backslash \Omega_{\delta/2}} N(v^\xi) \, u_\xi^\xi \, dx \right| \\ &\leqslant C\varepsilon^{-1} \, \|v^\xi\|^2_{L^2(\Omega_{\delta/2})} + e^{-c/\varepsilon} \, \|v^\xi\|^2_{L^2(\Omega)} \\ &\leqslant e^{-2\mathrm{d}^\xi/\varepsilon} e^{-c/\varepsilon}.\end{aligned}$$

Utilizing Lemma 3.4 and (1.5) we obtain by direct calculation

$$\begin{aligned}\varepsilon^2 \left| \int_{\partial\Omega} v^\xi \, \frac{\partial u_\xi^\xi}{\partial n} \, dS \right| &\leqslant C \left(\varepsilon^{2+\alpha^-} e^{-\xi/\varepsilon} \int_{\partial\Omega^-} \left| \frac{\partial u_\xi^\xi}{\partial n} \right| dS \right. \\ &\qquad \left. + \varepsilon^{2+\alpha^+} e^{-(1-\xi)/\varepsilon} \int_{\partial\Omega^+} \left| \frac{\partial u_\xi^\xi}{\partial n} \right| dS + e^{-2\mathrm{d}^\xi/\varepsilon} e^{-c/\varepsilon} \right) \\ &\leqslant C(\varepsilon^{1+2\alpha^-} e^{-2\xi/\varepsilon} + \varepsilon^{1+2\alpha^+} e^{-2(1-\xi)/\varepsilon}).\end{aligned}$$

Combining the last two inequalities and (3.9) yields (3.13). ∎

4. PROOF OF LEMMA 3.1

We begin by recalling the well known fact from the parabolic potential theory (see [21] for example).

LEMMA 4.1. *Let* $\Gamma(x, t; y, s)$ *be the fundamental solution of* $L^{\varepsilon, O}$ *and let* v^ξ *be a solution of* (2.8a, b, c). *Then*

$$\begin{aligned}v^\xi(x, t) = \int_0^t \int_{\partial\Omega} \Gamma(x, t; y, s) \, \phi(y, s) \, dS_y \, ds + \int_\Omega \Gamma(x, t; y, 0) \, v_0(y) \, dy \\ - \int_0^t \int_\Omega \Gamma(x, t; y, s) \, \Phi(y, s) \, dy \, ds,\end{aligned} \tag{4.1}$$

where $\phi(x, t)$, $(x, t) \in \partial\Omega \times (0, T_\delta)$ *is to be determined from the following Volterra equation*

$$\frac{1}{2}\phi(x, t) = \int_0^t \int_{\partial\Omega} \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \phi(y, s)\, dS_y\, ds + \frac{\partial u^\xi}{\partial n}(x, t)$$
$$- \int_0^t \int_\Omega \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \Phi(y, s)\, dy\, ds$$
$$+ \int_\Omega \frac{\partial \Gamma(x, t; y, 0)}{\partial n_x} v_0(y)\, dy. \tag{4.2}$$

In order to utilize the above representation we need to estimate the auxiliary function ϕ. The following version of Gronwall's inequality will be used.

LEMMA 4.2. *Let* Ω *be a bounded, open domain with* $\partial\Omega \in C^{1,\alpha}$. *Let* $\Gamma(x, t; y, s)$ *be as in Lemma* 4.1. *Assume that* $\psi(x, t)$, $\Psi(x, t)$, $(x, t) \in \partial\Omega \times \{t > 0\}$ *are nonnegative, continuous functions such that*

$$\psi(x, t) \leqslant \Psi(x, t) + 2 \int_0^t \int_{\partial\Omega} \left| \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \right| \psi(y, s)\, dS_y\, ds.$$

Then $\forall \theta \in (0, 1)$ *we have*

$$\psi(x, t) \leqslant \Psi(x, t) + \int_0^t \int_{\partial\Omega} D_{\alpha, \theta, \varepsilon}(x, t; y, s)\, \Psi(y, s)\, dS_y\, ds, \tag{4.3}$$

where

$$D_{\alpha, \theta, \varepsilon}(x, t; y, s) = e^{-(t-s)/\varepsilon^2} e^{-\theta |x-y|^2/4(t-s)}$$
$$\times \sum_{m=1}^{\infty} \frac{\Gamma^{m+1}(\alpha/2)\, \bar{K}^{m+1} (t-s)^{\alpha_m}}{\Gamma(\alpha_m + (N+1)/2)}. \tag{4.4a}$$

$\bar{K} = \bar{K}(\Omega, \theta)$ *is a positive constant and* $\alpha_m = -(N + 1 - m\alpha)/2$. *Moreover we have the estimate*

$$D_{\alpha, \theta, \varepsilon}(x, t; y, s) \leqslant \bar{K}_1 e^{-(t-s)/2\varepsilon^2} e^{-\theta |x-y|^2/4(t-s)} (t-s)^{-(N+1-\alpha)/2}, \tag{4.4b}$$

where $\bar{K}_1$ *is a positive constant.*

Proof of this lemma follows from standard calculations for the parabolic potentials, see for example [21] pp. 14–17. The details are omitted.

If we set

$$\begin{aligned}\Psi(x,t)=&\left|\frac{\partial u^{\xi}}{\partial n}(x,t)\right|+\int_0^t\int_{\partial\Omega}\left|\frac{\partial\Gamma(x,t;y,s)}{\partial n_x}\right|[|\Phi_1(y,s)|+|\Phi_2(y,s)|]\,dy\,ds\\&+\int_{\Omega}\left|\frac{\partial\Gamma(x,t;y,0)}{\partial n_x}\right||v(y,0)|\,dy\\&=\left|\frac{\partial u^{\xi}}{\partial n}(x,t)\right|+\sum_{i=1}^{3}\Psi_i,\end{aligned}\tag{4.5}$$

where

$$\Phi_1=\xi_t u^{\xi}_{\xi}+N(v^{\xi}),\qquad \Phi_2=\varepsilon^{-2}[F''(u^{\xi})-1]\,v^{\xi},$$

then from (4.2) it follows

$$|\phi(x,t)|\leqslant 2\int_0^t\int_{\partial\Omega}\left|\frac{\partial\Gamma(x,t;y,s)}{\partial n_x}\right||\phi(y,s)|\,dS_y\,ds+\Psi(x,t).$$

Hence Lemma 4.2 implies

$$\begin{aligned}|\phi(x,t)|\leqslant\,&\Psi(x,t)\\&+\bar{K}_1\int_0^t\int_{\partial\Omega}(t-s)^{(\alpha-1)/2}\,G_\theta(x,t;y,s)\,e^{-(t-s)/2\varepsilon^2}\Psi(y,s)\,dS_y\,ds,\end{aligned}\tag{4.6}$$

where

$$G_\theta(x,t;y,s)=(t-s)^{-N/2}\,e^{-\theta\,|x-y|^2/4(t-s)}.$$

We set

$$\Omega(a,b)=\{x\in\Omega\mid a<x_1<b\},\quad \Omega[a,b]=\overline{\Omega(a,b)},\quad \Omega^c(a,b)=\Omega\backslash\Omega(a,b).$$

LEMMA 4.3. *Let Ψ be the function defined in* (4.5). *There exist constants $c,\gamma>0$ depending on δ such that*

$$\Psi(x,t)\leqslant\begin{cases}(1+t^{-1/2})\,e^{-\mathrm{d}^{\xi}/\varepsilon}e^{-c/\varepsilon},\\ \quad \text{for}\quad x\in\{\Omega[\delta/8,7\delta/8]\cup\Omega[1-7\delta/8,1-\delta/8]\}\cap\partial\Omega,\\ \varepsilon^{-\gamma}(1+t^{-1/2}e^{-\delta/\varepsilon})\,e^{-\mathrm{d}^{\xi}/\varepsilon},\qquad \text{otherwise}.\end{cases}\tag{4.7}$$

Proof. We shall estimate various terms in (4.5). We have

$$\left|\frac{\partial u^{\xi}}{\partial n}(x,t)\right| \leqslant \begin{cases} C\varepsilon^{-1}e^{-\mathrm{d}^{\xi}/\varepsilon}, & \text{if} \quad x \notin \Omega[0,1] \cap \partial\Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

As in Lemma 2.4 we obtain

$$|\xi_t| \leqslant e^{-c/\varepsilon}e^{-\mathrm{d}^{\xi}/\varepsilon}, \qquad c>0. \tag{4.9a}$$

By applying Lemma 2.8 and Lemma 2.3 we obtain

$$\begin{aligned} \int_0^t \left| \int_\Omega N(v^\xi) \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \, dy \, ds \right| & \\ \leqslant C\varepsilon^{-2} \int_0^t \int_\Omega |v^\xi|^2 \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dy \, ds & \\ \leqslant C\varepsilon^{-2} \int_0^t \|v^\xi\|^2_{C^0(\Omega)} \int_\Omega \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dy \, ds & \\ \leqslant C\varepsilon^{-2} \int_0^t \|v^\xi\|^2_{C^0(\Omega)} \left\| \frac{\partial \Gamma(x,t;\cdot,s)}{\partial n_x} \right\|_{L^1(\Omega)} & \\ \leqslant e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \qquad c>0. & \end{aligned} \tag{4.9b}$$

From (4.9a, b) we then get by applying Lemma 2.3 again

$$\begin{aligned} \Psi_1(x,t) &= \int_0^t \int_\Omega |\Phi_1(y,s)| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dy \, ds \\ &= \int_0^t \int_\Omega |\xi_t u^\xi_\xi + N(v^\xi)| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dy \, ds \\ &\leqslant e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon} \qquad c, \gamma > 0. \end{aligned} \tag{4.9c}$$

For estimating Ψ_2 we first assume $x \notin \Omega[\delta/8, 7\delta/8] \cup \Omega[1-7\delta/8, 1-\delta/8]$. We then have

$$\begin{aligned} \Psi_2(x,t) &= \varepsilon^{-2} \int_0^t \int_\Omega |v^\xi[F''(u^\xi)-1]| \left| \frac{\partial \Gamma(x,t;y,s)}{\partial n_x} \right| dy \, ds \\ &\leqslant C\varepsilon^{-2} \int_0^t \|v^\xi\|_{C^0(\Omega)} \left\| \frac{\partial \Gamma(x,t;\cdot,s)}{\partial n_x} \right\|_{L^1(\Omega)} ds \\ &\leqslant \varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \end{aligned} \tag{4.10a}$$

Let's now assume $x \in \Omega[\delta/8, 7\delta/8]$. We can write

$$\Psi_2(x, t) = \int_0^t \left\{ \int_{\Omega_{\delta/16}} + \int_{\Omega \setminus \Omega_{\delta/16}} \right\} \varepsilon^{-2} \, |v^\xi [F''(u^\xi) - 1]| \left| \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \right| dy\, ds$$

$$=: \int_0^t [\Phi_{21}(x, t; s) + \Phi_{22}(x, t; s)] \, ds,$$

where $\Omega_{\delta/16} = \Omega(\delta/16, 15\delta/16)$.

For estimating $\Phi_{21}(x, t; s)$ we observe that if $y \in \Omega$ with $y_1 < 15\delta/16$ then we have $|F''(u^\xi) - 1| \leqslant Ce^{-\delta/16\varepsilon}$, and hence it follows

$$\int_0^t \Phi_{21}(x, t; s) \, ds \leqslant C\varepsilon^{-2} e^{-\delta/16\varepsilon} \int_0^t \int_\Omega |v^\xi| \left| \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \right| dy\, ds$$

$$\leqslant C\varepsilon^{-2} e^{-\delta/16\varepsilon} \int_0^t \|v^\xi\|_{C^0(\Omega)} \left\| \frac{\partial \Gamma(x, t; \cdot, s)}{\partial n_x} \right\|_{L^1(\Omega)} ds$$

$$\leqslant e^{-\delta/32\varepsilon} e^{-\mathrm{d}^{\xi(t)}/\varepsilon}. \tag{4.10b}$$

We notice that if $x \in \Omega[\delta/8, 7\delta/8] \cap \partial\Omega$, $y \in \Omega^c(\delta/16, 15\delta/16)$ then $|x - y| > \delta/16$. To estimate $\Phi_{22}(x, t; s)$ we write for any $\theta \in (0, 1)$

$$\int_0^t \Phi_{22}(x, t; s) \, ds$$

$$= \int_0^{t-2\varepsilon} \Phi_{22}(x, t; s) \, ds + \int_{t-2\varepsilon}^t \Phi_{22}(x, t; s) \, ds$$

$$\leqslant C\varepsilon^{-2} \left\{ \int_0^{t-2\varepsilon} \int_\Omega |v^\xi(y, s)| \left| \frac{\partial \Gamma(x, t; y, s)}{\partial n_x} \right| dy\, ds \right.$$

$$\left. + \int_{t-2\varepsilon}^t \int_{\Omega \cap \{|x-y| > \delta/16\}} |v^\xi(y, s)| \, e^{-\theta |x-y|^2/8\varepsilon} \left(\frac{\delta}{16} \right)^{-(N+1)} dy\, ds \right\}$$

$$\leqslant C\varepsilon^{-2} \int_0^{t-2\varepsilon} \|v^\xi(\cdot, s)\|_{C^0(\Omega)} \, (t - s)^{-1/2} \, e^{-(t-s)/\varepsilon^2} \, ds$$

$$+ C\varepsilon^{-\gamma} e^{-\mathrm{d}^{\xi(t)}/\varepsilon} \varepsilon^{-\theta\delta^2/2^{10}\varepsilon}$$

$$\leqslant e^{-1/\varepsilon} + e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-\theta\delta^2/2^{12}\varepsilon}. \tag{4.10c}$$

Finally from (2.8c) we get

$$\int_{\Omega} \left| \frac{\partial \Gamma(x, t; y, 0)}{\partial n_x} \right| |v(y, 0)| \, dy \leqslant t^{-1/2} e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}. \tag{4.11}$$

Combining (4.9c), (4.10a, b, c), (4.8), (4.11) we obtain (4.7). The proof is complete. ∎

COROLLARY 4.4. *Let ϕ be the auxiliary function defined in* (4.2). *There exist constants $c, \gamma > 0$ such that*

$$|\phi(x, t)| \leqslant \begin{cases} (1 + t^{-1/2}) \, e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-c/\varepsilon}, \\ \quad \textit{for } x \in \{\Omega[\delta/4, 3\delta/4] \cup \Omega[1 - 3\delta/4, 1 - \delta/4]\} \cap \partial\Omega, \\ \varepsilon^{-\gamma}(1 + t^{-1/2} e^{-\delta/\varepsilon}) \, e^{-\mathrm{d}^{\xi(t)}/\varepsilon}, \qquad \textit{otherwise}. \end{cases} \tag{4.12}$$

Proof. From Lemma 4.2 and Lemma 4.3 we see that it sufficies to consider

$$\begin{aligned} &\int_0^t \int_{\partial\Omega} (t - s)^{(\alpha - 1)/2} \, G_\theta(x, t; y, s) \, e^{-(t-s)/2\varepsilon^2} \, \Psi(y, s) \, dS_y \, ds \\ &\quad = \left(\int_0^{t - 6\varepsilon} + \int_{t - 6\varepsilon}^t \right) \int_{\partial\Omega} (t - s)^{(\alpha - 1)/2} \, G_\theta(x, t; y, s) \, e^{-(t-s)/2\varepsilon^2} \, \Psi(y, s) \, dS_y \, ds \\ &\quad =: I_1 + I_2. \end{aligned}$$

For estimating I_1 we make use of the fact that $e^{-(t-s)/2\varepsilon^2} \leqslant e^{-2/\varepsilon}$ if $t - s > 6\varepsilon$ hence

$$I_1 \leqslant e^{-1/\varepsilon} \leqslant e^{-\mathrm{d}^{\xi(t)}/\varepsilon} e^{-\delta'/\varepsilon}.$$

To estimate I_2 we use (4.7) and argue as in (4.10a, b, c) of the previous lemma. The proof of the corollary is complete. ∎

Proof of Lemma 4.1. Lemma 4.1 follows by the argument similiar to the one in the proof of Lemma 4.3. Formula (4.1) is used instead of (4.2) to derive the estimate analogous to (4.6). Details are omitted. ∎

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